Q. Solve the Initial-Value Problem

$$\dot{x} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} x \qquad x(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution:

Eigenvalues of $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 3 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)^2 - 0\} = 0$$

$$\Rightarrow (2-\lambda)^3 = 0$$

ie. $\lambda = 2, 2, 2$

ie $\lambda = 2$ is an eigenvalue of A with multiplicity three.

Eigenvector corresponding to eigenvalue $\lambda = 2$

$$[A - 2I]v = 0$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_2 + 3v_3 = 0$$
$$-v_3 = 0.$$

$$\Rightarrow \quad v_2 = 0, \quad v_3 = 0$$

and v_1 is arbitrary.

Hence $$x'(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is one solution of given diff. eq^n.

Now,

$$(A-2I)^2 v = 0$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $v_3 = 0$

and both v_1 and v_2 are arbitrary

But $(A-2I)v \neq 0$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $v_2 + 3v_3 \neq 0$

$v_3 \neq 0.$

$\Rightarrow$ $v_2 \neq 0, \quad v_3 \neq 0.$

$\therefore$ We take,

$$v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad x^2(t) = e^{\lambda t}\left[I + \frac{(A-\lambda I)t}{1!} \right] v$$

$$= e^{2t}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} t \right\} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} 1 & t & 3t \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

Which is a second L.I. solution of given diff. eq^n.

for the third solution

$$(A-2I)^3 v = 0$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_1 = 0 = v_2 = v_3$$

but $(A-2I)^2 v \neq 0$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq 0$$

$\Rightarrow$ $v_3 \neq 0$ v_1 & v_2 arb

$\therefore$ we take, $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$x^3(t) = e^{\lambda t}\left[I + (A-\lambda I)t + (A-\lambda I)^2\frac{t^2}{2!}\right]$$

$$= e^{2t}\left[I + (A-2I)t + (A-2I)^2\frac{t^2}{2!}\right]$$

$$= e^{2t}\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}t + \begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\frac{t^2}{2!}\right\}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= e^{2t}\begin{bmatrix}1&t&3t-\frac{t^2}{2}\\0&1&-t\\0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= e^{2t}\begin{bmatrix}3t-\frac{t^2}{2}\\-t\\1\end{bmatrix}$$

Which is third L.I. sol^n of given diff eq^n

$\therefore \cancel{x(t) = c_1 e^{2t} + c_2 e^{3t} + c_3 e^{2t}}$

$$\therefore\ x(t) = c_1 x^1(t) + c_2 x^2(t) + c_3 x^3(t).$$

$$= e^{2t}\left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3t - \frac{t^2}{2} \\ -t \\ 1 \end{bmatrix} \right.$$

putting $t = 0$.

$$x(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow c_1 = 2,\ c_2 = 2,\ c_3 = 1$$

$$\therefore\ x(t) = e^{2t} \begin{bmatrix} 2 + 2t + 3t - \frac{t^2}{2} \\ 2 - t \\ 1 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} 2 + 5t - \frac{t^2}{2} \\ 2 - t \\ 1 \end{bmatrix}$$